

SINC INTEGRALS AND TINY NUMBERS

UWE BÄSEL AND ROBERT BAILLIE

ABSTRACT. We apply a result of David and Jon Borwein to evaluate a sequence of highly-oscillatory integrals whose integrands are the products of a rapidly growing number of sinc functions. The value of each integral is given in the form $\pi(1-t)/2$, where the numbers t quickly become very tiny. Using the Euler-Maclaurin summation formula, we calculate these numbers to high precision. For example, the integrand of the tenth integral in the sequence is the product of 68100152 sinc functions. The corresponding t is approximately

$$9.6492736004286844634795531209398105309232 \cdot 10^{-554381308}.$$

1. INTRODUCTION

Leonhard Euler knew that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \quad (1)$$

at the latest by 1781 [15, p. 324], and there exist several proofs for it (see, for example [8], [15, p. 324], and, for a proof due to Lobachevsky, [7, pp. 635-636]). The substitution $x \mapsto a_0 x$ with a real number a_0 immediately shows, more generally, that (see also [13, pp. 83-84])

$$I_0 = \int_0^\infty \frac{\sin(a_0 x)}{x} dx = \frac{\pi}{2} \quad \text{if } a_0 > 0. \quad (2)$$

Pólya [14, pp. 208-209] and D. & J. Borwein [6, pp. 78-79] derived in different ways the general evaluation of the integral

$$I_n = \int_0^\infty \frac{\sin(a_0 x)}{x} \frac{\sin(a_1 x)}{x} \dots \frac{\sin(a_n x)}{x} dx \quad (3)$$

where a_0, a_1, \dots, a_n are real numbers. If, in addition, $a_0, a_1, \dots, a_n > 0$ with

$$a_0 \geq s(n) := \sum_{k=1}^n a_k, \quad (4)$$

then the general solution simplifies to

$$I_n = \frac{\pi}{2} a_1 a_2 \dots a_n, \quad (5)$$

see [7, p. 654] and [6, pp. 78-79]. Furthermore, Corollary 1 of [6] (see also Theorem 2 of [3]) says that, if

$$2a_k \geq a_n > 0 \quad \text{for } k = 0, 1, \dots, n-1, \quad (6)$$

2010 *Mathematics Subject Classification.* Primary 33B10; Secondary 26D15, 33F05.

Key words and phrases. sinc function, sinc integrals, small numbers, Euler-Maclaurin summation formula, Stirling's approximation.

and n is such that the sum of $a_1 + a_2 + \dots + a_n$ first exceeds a_0 ,

$$s(n) > a_0 \geq s(n-1), \quad (7)$$

then we have this formula for the exact value of the integral:

$$I_n = \frac{\pi}{2} \left\{ \prod_{k=1}^n a_k - \frac{(a_1 + a_2 + \dots + a_n - a_0)^n}{2^{n-1} n!} \right\}. \quad (8)$$

Now we consider the integral

$$J_n = a_0 \int_0^\infty \prod_{k=0}^n \text{sinc}(a_k x) dx = \frac{I_n}{a_1 a_2 \dots a_n} \quad (9)$$

where the sinc function is defined as

$$\text{sinc}(x) = \begin{cases} \sin(x)/x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Eq. (2) yields

$$J_0 = a_0 \int_0^\infty \text{sinc}(a_0 x) dx = \frac{\pi}{2} \quad \text{if } a_0 > 0. \quad (10)$$

For $n = 1, 2, \dots$, and $a_0, a_1, \dots, a_n > 0$, from (4), (5), (6), (7), (8), and (9) it follows that

$$J_n = \begin{cases} \frac{\pi}{2} & \text{if } a_0 \geq s(n), \\ \frac{\pi}{2}(1 - t_n) & \text{if (6) and (7) hold,} \end{cases} \quad (11)$$

$$(12)$$

where

$$t_n = \frac{(a_1 + a_2 + \dots + a_n - a_0)^n}{2^{n-1} n! \prod_{k=1}^n a_k}. \quad (13)$$

Using Theorem 1 (ii) of [6], we see that

$$0 < J_{n+1} \leq J_n < \pi/2 \quad \text{if } a_{n+1} \leq a_0 < s(n), \quad n \geq 1. \quad (14)$$

Schmid [17, pp. 13-16] proves that $J_{n+1} < J_n < \pi/2$ if $\{a_k\}$ is a monotonically non-increasing series of positive real numbers with $a_0 < s(n)$.

In our applications below, the a_k are defined as

$$a_0 \text{ is an integer } \geq 1, \quad a_k = \frac{1}{2k-1} \quad \text{for } k = 1, 2, \dots, n, \quad (15)$$

and we write $J_n(a_0)$ and $t_n(a_0)$ instead of J_n and t_n , respectively.

From (11) and (14) we know (see also [1, pp. 3-4], with a slight change in notation) that

$$J_n(a_0) = \frac{\pi}{2} \quad \text{if } a_0 \geq s(n) = \sum_{k=1}^n \frac{1}{2k-1}, \quad (16)$$

$$J_n(a_0) < \frac{\pi}{2} \quad \text{if } a_0 < s(n). \quad (17)$$

It is easy to see that our a_k as defined in (15) satisfy the inequalities (6): our a_k are all positive with $a_k \geq a_n$ for $k = 0, 1, \dots, n-1$, which implies that $2a_k \geq a_n$. If, in addition, the inequalities (7) are satisfied, then we are able to compute the *exact* value of $J_n(a_0)$, given by Equations (12)

and (13). $t_n(a_0)$ can be a *very* tiny number. *Our main aim in this paper is to show how to calculate these tiny numbers with high precision.*

With *Mathematica* we can calculate a few of these integrals directly. For example,

$$J_2(1) = \int_0^\infty \text{sinc}(x) \text{sinc}\left(\frac{x}{1}\right) \text{sinc}\left(\frac{x}{3}\right) dx = \frac{11}{24} \pi \approx 0.458333 \pi.$$

In this example, we have $a_0 = 1$, $a_1 = 1$, and $a_2 = 1/3$. Note that

$$a_1 \leq a_0 < s(2) = a_1 + a_2 = \frac{4}{3},$$

so using the inequalities (7), we have $n = 2$. Eq. (10) tells us that

$$J_0(1) = \int_0^\infty \text{sinc}(x) dx = \frac{\pi}{2},$$

and Eq. (11) delivers

$$J_1(1) = \int_0^\infty \text{sinc}(x) \cdot \text{sinc}(x) dx = \frac{\pi}{2}.$$

For the equation $J_1(1) = J_0(1)$ see, for example, [2], [3], [5], [15, p. 324]. From Eq. (12) we get the already known result

$$J_2(1) = \frac{\pi}{2} \left\{ 1 - \frac{(\frac{4}{3} - 1)^2}{2^1 \cdot 2! \cdot \frac{1}{1} \cdot \frac{1}{3}} \right\} = \frac{11\pi}{24}.$$

As we include more sinc functions in the integrand, it generally takes more time for *Mathematica* to evaluate the integral. *Mathematica* is able to calculate that

$$J_n(2) = 2 \int_0^\infty \text{sinc}(2x) \text{sinc}\left(\frac{x}{1}\right) \cdots \text{sinc}\left(\frac{x}{2n-1}\right) dx = \frac{\pi}{2}$$

for all $n = 1, 2, \dots, 7$.

Mathematica is also able to calculate (see also [1, p. 4])

$$\begin{aligned} J_8(2) &= 2 \int_0^\infty \text{sinc}(2x) \text{sinc}\left(\frac{x}{1}\right) \text{sinc}\left(\frac{x}{3}\right) \cdots \text{sinc}\left(\frac{x}{15}\right) dx \\ &= \frac{168579263752211300739165075916829279}{337158527504429357358419617830000000} \pi \approx 0.49999999999998998115 \pi. \end{aligned}$$

We can also evaluate this integral using Equations (12) and (13). For $J_8(2)$, we have $a_0 = 2$, $a_1 = 1$, $a_2 = 1/3$, \dots , $a_8 = 1/15$. Then

$$\sum_{k=1}^7 a_k = \frac{88069}{45045} = 1.95513 \dots < a_0 < 2.02181 \dots = \frac{91072}{45045} = \sum_{k=1}^8 a_k,$$

and therefore,

$$\begin{aligned} J_8(2) &= \frac{\pi}{2} \left\{ 1 - \frac{(\frac{91072}{45045} - 2)^8}{2^7 \cdot 8! \cdot \frac{1}{1} \cdot \frac{1}{3} \cdots \frac{1}{15}} \right\} = \frac{\pi}{2} \left\{ 1 - \frac{3377940044732998170721}{168579263752214678679209808915000000} \right\} \\ &= \frac{168579263752211300739165075916829279}{337158527504429357358419617830000000} \pi. \end{aligned}$$

$S + 1$. The terms in our series,

$$\sum_{k=1}^n \frac{1}{2k-1},$$

are about $1/2$ as large as the corresponding terms in the harmonic series. Therefore, to increase our sum by 1 requires about as many terms as the harmonic series needs to increase its sum by 2, which is about $e \cdot e = e^2$.

As noted above, Eq. (12) gives the *exact* value of the integral. The expression on the right side of Eq. (12) may be written as

$$\frac{\pi}{2} \left(1 - \frac{P}{Q} \right),$$

where P and Q are integers. As a_0 increases, P and Q quickly become very large. For example, with $a_0 = 6$, P and Q have 453130185 and 453237210 digits, respectively. Displaying the first and last 20 digits for this case, we have

$$\frac{P}{Q} = \frac{34293043773392420460 \text{ (453130145 digits) } 34573721229967337961}{26251415654224851611 \text{ (453237170 digits) } 00000000000000000000}.$$

3. A NOTE ON PRECISION

Eq. (12) requires that we first compute the sum $s(n)$, then raise $s(n) - a_0$ to a very high power. For example, with $a_0 = 7$, we have $n = 168804$ and $s(168804) \approx 7 + 1.79178 \cdot 10^{-6}$. We then compute $(s(168804) - 7)^{168804}$. Many, or even *all*, of the significant digits of $s(n)$ will be lost if we calculate $s(n)$ to only machine precision. Therefore, we did our calculations twice: first, we computed each $s(n)$ to 60 decimals, then used this value in Equation (13). Then, we repeated the calculations, this time, computing each $s(n)$ to 70 decimals. These high-precision results agree with each other to more decimals than we show in Table 1. On the other hand, for $a_0 = 7$, only the first *three* digits of the machine precision calculation agree with these high-precision results. Worse, when we do the calculation for $a_0 = 8$ in machine precision, we get, approximately,

$$\frac{\pi}{2} (1 - 1.03496 \cdot 10^{-8742942}).$$

Note that *all digits and the exponent* are different from the high-precision result in Table 1.

4. APPLYING THE EULER-MACLAURIN SUMMATION FORMULA

In our special case where $a_k = 1/(2k-1)$ for $k \geq 1$, we can use estimates of partial sums of the harmonic series to estimate $s(n) = \sum_{k=1}^n a_k$. Define H_N to be the N^{th} partial sum of the harmonic series

$$H_N = \sum_{k=1}^N \frac{1}{k}.$$

Notice that

$$\sum_{k=1}^n a_k = \sum_{k=1}^n \frac{1}{2k-1} = H_{2n-1} - \frac{1}{2}H_{n-1}. \quad (18)$$

H_N has the asymptotic approximation (see [19])

$$H_N \asymp \ln(N) + \gamma + \frac{1}{2N} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kN^{2k}} = \ln(N) + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + \frac{1}{120N^4} - \dots$$

which is proved (see [9, p. 78]) using the Euler-Maclaurin summation formula. We can use this in Eq. (18) to get a good approximation to $s(n) = \sum_{k=1}^n a_k$.

Here, we show how to use the Euler-Maclaurin summation formula to calculate exactly the smallest n for which the sum of the a_k exceeds a_0 . We also use the Euler-Maclaurin summation formula to calculate the sum of the a_k for any large n . This method applies to general a_k , and gives us an estimate of the error.

One version of the Euler-Maclaurin summation formula is (see e.g. [10, pp. 542-543])

$$\begin{aligned} \sum_{k=m}^n f(k) &= \int_m^n f(x) dx + \frac{f(m) + f(n)}{2} \\ &\quad + \sum_{j=1}^{\mu} \frac{B_{2j}}{(2j)!} (f^{(2j-1)}(n) - f^{(2j-1)}(m)) + R_{\mu}(m, n) \end{aligned} \quad (19)$$

with the remainder term

$$\begin{aligned} R_{\mu}(m, n) &= \int_m^n \frac{B_{2\mu+1}(x - \lfloor x \rfloor)}{(2\mu+1)!} f^{(2\mu+1)}(x) dx \\ &= \frac{1}{(2\mu+1)!} \sum_{k=m}^{n-1} \int_0^1 B_{2\mu+1}(x) f^{(2\mu+1)}(k+x) dx. \end{aligned} \quad (20)$$

$B_k(x)$ denotes the k^{th} Bernoulli polynomial, and $B_k = B_k(0)$ the k^{th} Bernoulli number. In our case we have

$$f(x) = \frac{1}{2x-1} \quad \text{and} \quad a_k = f(k) = \frac{1}{2k-1}.$$

Now we will derive an estimate of $R_{\mu}(m, n)$. For the k^{th} derivative of f , one finds

$$f^{(k)}(x) = \frac{(-1)^k 2^k k!}{(2x-1)^{k+1}}.$$

Since all the functions $|f^{(k)}(x)|$, $k = 0, 1, 2, \dots$, are strictly decreasing, for the terms in the sum of (20) we find

$$\left| \int_0^1 B_{2\mu+1}(x) f^{(2\mu+1)}(k+1+x) dx \right| < \left| \int_0^1 B_{2\mu+1}(x) f^{(2\mu+1)}(k+x) dx \right|.$$

The absolute value of each integral on the right-hand side of Equation (20) is at most

$$\left| \int_0^1 B_{2\mu+1}(x) f^{(2\mu+1)}(m+x) dx \right|$$

and there are $n - m$ of these integrals. Therefore

$$|R_{\mu}(m, n)| < |\widetilde{R}_{\mu}(m, n)| \quad (21)$$

where

$$\tilde{R}_\mu(m, n) = \frac{n-m}{(2\mu+1)!} \int_0^1 B_{2\mu+1}(x) f^{(2\mu+1)}(m+x) dx.$$

Equation (21) is the desired estimate for $R_\mu(m, n)$. Furthermore, all

$$\int_0^1 B_{2\mu+1}(x) f^{(2\mu+1)}(k+x) dx, \quad k = m, \dots, n-1,$$

have the same sign which is equal to the sign of $R_\mu(m, n)$, and to the sign of $\tilde{R}_\mu(m, n)$.

Using the integral

$$\int_m^n f(x) dx = \int_m^n \frac{dx}{2x-1} = \frac{1}{2} [\ln(2n-1) - \ln(2m-1)],$$

we get the explicit summation formula

$$\sum_{k=m}^n \frac{1}{2k-1} = \varphi_\mu(m, n) + R_\mu(m, n) \quad (22)$$

with the approximation

$$\begin{aligned} \varphi_\mu(m, n) = & \frac{1}{2} \left(\ln(2n-1) - \ln(2m-1) + \frac{1}{2m-1} + \frac{1}{2n-1} \right) \\ & - \sum_{j=1}^{\mu} \frac{2^{2j-1} B_{2j}}{2j} \left(\frac{1}{(2n-1)^{2j}} - \frac{1}{(2m-1)^{2j}} \right) \end{aligned} \quad (23)$$

and the remainder term

$$R_\mu(m, n) = -2^{2\mu+1} \sum_{k=m}^{n-1} \int_0^1 \frac{B_{2\mu+1}(x)}{[2(k+x)-1]^{2\mu+2}} dx.$$

The explicit formula for the error bound is

$$\tilde{R}_\mu(m, n) = -2^{2\mu+1} (n-m) \int_0^1 \frac{B_{2\mu+1}(x)}{[2(m+x)-1]^{2\mu+2}} dx. \quad (24)$$

4.1. Calculating the n That Satisfies Inequalities (7).

Our first application of Eq. (19) is to calculate the value of $n = n_0$ that satisfies (7) for a fixed integer value of a_0 . (We can calculate the integrals for $a_0 < 10$ without too much trouble, so here, we are interested in the values $a_0 \geq 10$.) Writing the sum $s(n)$ as

$$s(n) = \sum_{k=1}^n \frac{1}{2k-1} = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} = \frac{1}{1} + \frac{1}{1+1 \cdot 2} + \frac{1}{1+2 \cdot 2} + \dots + \frac{1}{1+(n-1)2},$$

from a theorem of Nagell [12, pp. 10-14] (see also [4]) it easily follows that $s(n)$ is never an integer except $s(1) = 1$. So we can replace (7) by

$$s(n-1) < a_0 < s(n). \quad (25)$$

Using Eq. (22), we have

$$s(n) = s(m-1) + \varphi_\mu(m, n) + R_\mu(m, n). \quad (26)$$

Hence an approximation for $s(n)$ is

$$\tilde{s}_{m,\mu}(n) = s(m-1) + \varphi_\mu(m, n). \quad (27)$$

The error bound for the sum

$$\sum_{k=m}^n \frac{1}{2k-1}$$

is given by

$$|\tilde{s}_{m,\mu}(n) - s(n)| < |\tilde{R}_\mu(m, n)|. \quad (28)$$

Eq. (24) shows that a larger m (with n and μ fixed) makes the error bound smaller. Therefore, we begin by explicitly computing the sum of the first $m-1$ terms,

$$s(m-1) = \sum_{k=1}^{m-1} \frac{1}{2k-1}$$

to high precision. This will be used to achieve the required precision of $\tilde{s}_{m,\mu}(n)$.

Replacing the integer variable n in Eq. (27) by the real variable x , we get the equation

$$\tilde{s}_{m,\mu}(x) = s(m-1) + \varphi_\mu(m, x) \quad (29)$$

with parameters m and μ . Solving

$$s(m-1) + \varphi_\mu(m, x) = a_0 \quad (30)$$

gives a value of x that approximates n_0 . Call this root r .

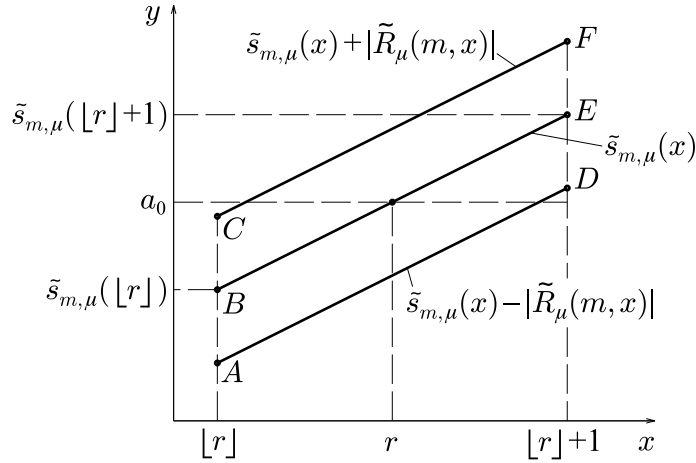


FIGURE 1. Checking if $[r] + 1 = n_0$

Now we must find a criterion that allows us to check if $[r] + 1$ is the value n_0 that satisfies (25). To this purpose we consider Fig. 1 with the graphs of the functions

$$\tilde{s}_{m,\mu}(x) + |\tilde{R}_\mu(m, x)|, \quad \tilde{s}_{m,\mu}(x), \quad \tilde{s}_{m,\mu}(x) - |\tilde{R}_\mu(m, x)|,$$

where (cf. Eq. (24))

$$\tilde{R}_\mu(m, x) = -2^{2\mu+1} (x - m) \int_0^1 \frac{B_{2\mu+1}(t)}{[2(m+t)-1]^{2\mu+2}} dt. \quad (31)$$

Now we distinguish the following two cases:

- a) $\tilde{R}_\mu(m, x) > 0$: From (28) it follows that the point $(\lfloor r \rfloor, s(\lfloor r \rfloor))$ is a point of the line segment \overline{BC} , and $(\lfloor r \rfloor + 1, s(\lfloor r \rfloor + 1))$ is a point of the line segment \overline{EF} . If, where $|\overline{BC}|$ denotes the length of \overline{BC} ,

$$|\overline{BC}| = \tilde{s}_{m,\mu}(\lfloor r \rfloor) + |\tilde{R}_\mu(m, \lfloor r \rfloor)| - \tilde{s}_{m,\mu}(\lfloor r \rfloor) < a_0 - \tilde{s}_{m,\mu}(\lfloor r \rfloor), \quad (32)$$

we know that $s(\lfloor r \rfloor) < a_0 < s(\lfloor r \rfloor + 1)$, hence $n_0 = \lfloor r \rfloor + 1$. Using (29) we write the inequality in (32) in the final form

$$a_0 - [s(m-1) + \varphi_\mu(m, \lfloor r \rfloor)] > |\tilde{R}_\mu(m, \lfloor r \rfloor)|. \quad (33)$$

- b) $\tilde{R}_\mu(m, x) < 0$: We have $(\lfloor r \rfloor, s(\lfloor r \rfloor)) \in \overline{AB}$ and $(\lfloor r \rfloor + 1, s(\lfloor r \rfloor + 1)) \in \overline{DE}$. If

$$|\overline{DE}| = \tilde{s}_{m,\mu}(\lfloor r \rfloor + 1) - [\tilde{s}_{m,\mu}(\lfloor r \rfloor + 1) - |\tilde{R}_\mu(m, \lfloor r \rfloor + 1)|] < \tilde{s}_{m,\mu}(\lfloor r \rfloor + 1) - a_0, \quad (34)$$

we know that $s(\lfloor r \rfloor) < a_0 < s(\lfloor r \rfloor + 1)$, hence $n_0 = \lfloor r \rfloor + 1$. The inequality in (34) may be written as

$$s(m-1) + \varphi_\mu(m, \lfloor r \rfloor + 1) - a_0 > |\tilde{R}_\mu(m, \lfloor r \rfloor + 1)|. \quad (35)$$

Inequalities (33) and (35) can be combined together into

$$|s(m-1) + \varphi_\mu(m, n) - a_0| > |\tilde{R}_\mu(m, n)| \quad (36)$$

with

$$n = \begin{cases} \lfloor r \rfloor & \text{if } \tilde{R}_\mu(m, x) > 0, \\ \lfloor r \rfloor + 1 & \text{if } \tilde{R}_\mu(m, x) < 0. \end{cases}$$

As an example we will calculate n_0 for $a_0 = 10$. This allows us to check our result against Table 1. From Table 1, it is clear that, for $a_0 \geq 10$, n_0 is at least several million, so we choose $m = 100001$ and find

$$s(m-1) = s(100000) \approx 6.73821774549790928310.$$

In this example, we compute this sum to 20 decimal places, but it is easy to compute more. We choose $\mu = 3$ in Eq. (27). For larger a_0 , or to obtain even more decimals of the sum of the a_k that exceeds $a_0 = 10$, it might be necessary to use a larger m , to compute $s(m-1)$ to more decimals, or to use a larger value of μ (or all of the above).

We then use *Mathematica's* `FindRoot` function to solve the equation

$$s(100000) + \varphi_3(100001, x) = 10$$

for x . Expanding $\varphi_3(m, x)$, we get

$$\begin{aligned} \varphi_3(m, x) = & \frac{1}{2} \left(\frac{1}{2m-1} - \ln(2m-1) + \frac{1}{2x-1} + \ln(2x-1) \right) - \frac{1}{6} \left(\frac{1}{(2x-1)^2} - \frac{1}{(2m-1)^2} \right) \\ & - \frac{8}{63} \left(\frac{1}{(2x-1)^6} - \frac{1}{(2m-1)^6} \right) + \frac{1}{15} \left(\frac{1}{(2x-1)^4} - \frac{1}{(2m-1)^4} \right). \end{aligned}$$

If we substitute $m = 100001$ and combine the numeric terms together into a decimal value, we get

$$\begin{aligned} \varphi_3(100001, x) = \\ -6.10303632276717019809 + \frac{1}{4x-2} - \frac{1}{6(1-2x)^2} + \frac{1}{15(1-2x)^4} - \frac{8}{63(1-2x)^6} + \frac{1}{2} \ln(2x-1) \end{aligned}$$

So, the equation we want to solve is

$$\begin{aligned} -6.10303632276717019809 + \frac{1}{4x-2} - \frac{1}{6(1-2x)^2} + \frac{1}{15(1-2x)^4} - \frac{8}{63(1-2x)^6} + \frac{1}{2} \ln(2x-1) \\ = 10 - 6.73821774549790928310 = 3.26178225450209071690. \end{aligned} \quad (37)$$

We find $x = r = 68100150.0149$. The signed error bound is found by numerical integration of Eq. (24):

$$\tilde{R}_3(100001, 68100150.0149) \approx -1.13323 \cdot 10^{-39}.$$

Since this error bound is less than 0, we use Eq. (36) with $n = \lfloor r \rfloor + 1 = 68100151$ in order to check if $n_0 = \lfloor r \rfloor + 1$. One finds that

$$|s(100000) + \varphi_3(100001, 68100151) - 10| \approx 7.23308281312 \cdot 10^{-9},$$

so the condition (36) holds true, hence $n_0 = 68100151$. This confirms the $n = 68100151$ in Table 1 that was found by brute force.

Once we know the value of $n = n_0$ for which the sum $s(n_0)$ first exceeds $a_0 = 10$, we must compute $s(n_0)$, which is used in Equations (12) and (13) to compute the value of the sinc integral. For example, with $a_0 = 10$, we find that $n_0 = 68100151$ and

$$s(n_0) \approx s(100000) + \varphi_3(100001, n_0) \approx 10.00000000723308281312.$$

Equation (13) requires that we raise the difference $s(n_0) - a_0$ to the high power n_0 . (Note the loss of precision that occurs when we perform this subtraction). So, we may need to compute more accurate approximations $\varphi_\mu(m, n_0)$ using values of $\mu > 3$. Tables 2 and 3 below display n_0 and the approximate values of $s(n_0)$ for $a_0 = 10, 11, \dots, 25$. To obtain these values, we use $m = 100001$ and compute $s(m-1)$ to 100 decimal places, then use $\mu = 10$ to compute each n_0 and $\varphi_\mu(m, n_0)$. The *Mathematica* module `getNValueAndSumForA0[]` in Appendix A.1 performs these calculations.

Note that if we compute the initial sum $s(m-1)$ to only D decimal places, then we can never compute $s(j)$ to more than D correct decimal places for any $j > m-1$, *even if* the error estimate $|\tilde{R}_\mu|$ is less than 10^{-D} .

5. CALCULATING THE INTEGRALS

For a given a_0 , we first compute the corresponding value of $n = n_0$ and the approximate value of $s(n)$, as shown in Tables 2 and 3. The next task is to compute the value of $J_n(a_0)$ using Equations (12) and (13). The value of $(s(n) - a_0)^n$ can easily be obtained from the approximate value of $s(n)$ in Table 3, although for large n , we must use logarithms to prevent underflow.

a_0	n
10	68100151
11	503195829
12	3718142208
13	27473561358
14	203003686106
15	1500005624924
16	11083625711271
17	81897532160125
18	605145459495141
19	4471453748222757
20	33039822589391676
21	244133102611731231
22	1803913190804074904
23	13329215764452299411
24	98490323038288832267
25	727750522131718025058

TABLE 2. Values of $n = n_0$, for each a_0

a_0	$\tilde{s}_{100001,10}(n)$	$\tilde{R}_{10}(100001, n)$
10	$10 + 7.23308281311740815495440938881892875629793229610802275303838659 \cdot 10^{-9}$	$2.10 \cdot 10^{-104}$
11	$11 + 1.93429694721571938243592220609208607459386666993511996115170447 \cdot 10^{-10}$	$1.56 \cdot 10^{-103}$
12	$12 + 2.81704757017003986061562163359221047582420212335754506062428273 \cdot 10^{-11}$	$1.15 \cdot 10^{-102}$
13	$13 + 1.51784528343340657974855459172890208869659078869207257172632024 \cdot 10^{-11}$	$8.50 \cdot 10^{-102}$
14	$14 + 1.20004359101609629122080445652372955218041261117217502496026183 \cdot 10^{-12}$	$6.28 \cdot 10^{-101}$
15	$15 + 2.19180272149887470909606761989402891098093034885435056479463010 \cdot 10^{-13}$	$4.64 \cdot 10^{-100}$
16	$16 + 4.03776701092542650935062088404145888641302878626731173386452500 \cdot 10^{-14}$	$3.43 \cdot 10^{-99}$
17	$17 + 3.96811610610919880012621568968292119992007054389850812439945895 \cdot 10^{-16}$	$2.54 \cdot 10^{-98}$
18	$18 + 6.44184629552359167120616511513071089671769035954843683552410240 \cdot 10^{-16}$	$1.87 \cdot 10^{-97}$
19	$19 + 4.40658835470283585071853820223887285629968834008507731684755191 \cdot 10^{-17}$	$1.38 \cdot 10^{-96}$
20	$20 + 2.90258683104894913499203070153167600669577064916856020926287204 \cdot 10^{-18}$	$1.02 \cdot 10^{-95}$
21	$21 + 7.34280669057054306832818424563959102068261955548079016234613646 \cdot 10^{-19}$	$7.56 \cdot 10^{-95}$
22	$22 + 1.30683560567708459204537388912129731492458474471888171662329379 \cdot 10^{-19}$	$5.58 \cdot 10^{-94}$
23	$23 + 3.40633844408955109014083203224199839911999656758748815953125439 \cdot 10^{-20}$	$4.13 \cdot 10^{-93}$
24	$24 + 3.79499658486046318316555581259771062170888781423675014763472623 \cdot 10^{-21}$	$3.05 \cdot 10^{-92}$
25	$25 + 1.14325480646582051223669818246654129326458197624224895807362049 \cdot 10^{-22}$	$2.25 \cdot 10^{-91}$

TABLE 3. $a_1 + a_2 + \dots + a_n$ (n from Table 2), and error bound from (24), for each a_0

Equation (13) requires that we compute the product of a_k for $k = 1, \dots, n$. In our examples, we have $a_k = 1/(2k - 1)$ for $k = 1, \dots, n$, so we can rewrite this product as follows:

$$\prod_{k=1}^n a_k = \prod_{k=1}^n \frac{1}{2k-1} = \frac{1}{2n-1} \cdot \frac{1}{2n-3} \cdots \frac{1}{3} \cdot \frac{1}{1}.$$

We can express this in terms of factorials:

$$\frac{1}{2n-1} \cdot \frac{1}{2n-3} \cdots \frac{1}{3} \cdot \frac{1}{1} = \frac{2n \cdot (2n-2) \cdots 4 \cdot 2}{2n \cdot (2n-1) \cdot 2 \cdot 1} = \frac{2^n \cdot n \cdot (n-1) \cdots 2 \cdot 1}{2n \cdot (2n-1) \cdot 2 \cdot 1} = \frac{2^n n!}{(2n)!}.$$

Therefore, we can rewrite Equations (12) and (13) as

$$J_n(a_0) = \frac{\pi}{2} \{1 - t_n(a_0)\} \quad \text{where} \quad t_n(a_0) = \frac{(s(n) - a_0)^n}{2^{2n-1}} \cdot \frac{(2n)!}{n!^2} \quad (38)$$

When n is as large as some of the values in Table 2, we will need to use logarithms. Taking natural logarithms, we have

$$\ln t_n(a_0) = n \ln(s(n) - a_0) - (2n-1) \ln 2 + \ln((2n)!) - 2 \ln(n!). \quad (39)$$

We denote by \lg the the log base 10 and have

$$\lg t_n(a_0) = \frac{\ln t_n(a_0)}{\ln(10)} \quad (40)$$

with $t_n(a_0)$ from Eq. (39). Now, it remains to estimate the logarithms of the factorials in Eq. (39). We will discuss two techniques: approximations based on Stirling's formula, and another application of the Euler-Maclaurin summation formula.

5.1. Estimating $n!$ With Stirling's Formula.

For large n , we can use Stirling's formula to get reasonable approximations to the above combination of factorials. These approximations mainly require a few exponentiations, which is much faster than doing $O(n)$ multiplications, especially when $n > 10^6$.

The simple version of Stirling's formula is

$$n! \approx \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}.$$

Strictly speaking, this is not an approximation to $n!$, but is an asymptotic relation. This means that, as n approaches ∞ , the *ratio* of the right-hand side to $n!$ approaches 1. However, for the large n in Table 3, the ratio is 1 to several decimal places, so we can obtain several decimal places of the right side in $\pi[1 - t_n(a_0)]/2$.

A more rigorous approach is to compute both lower and upper bounds for $n!$, based on refined versions of Stirling's formula. These bounds were proved in [16]; see also [18]:

$$\left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \cdot e^{\frac{1}{12n+1}} < n! < \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \cdot e^{\frac{1}{12n}}.$$

The following improved lower bound was proved in [11], so we will use

$$\left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \cdot e^{\frac{1}{12n + \frac{3}{4n+2}}} < n! < \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \cdot e^{\frac{1}{12n}}. \quad (41)$$

How close are the lower and upper bounds in (41)? The ratio of the upper bound to the lower bound is

$$r_n = \exp\left(\frac{1}{12n} - \frac{1}{12n + \frac{3}{4n+2}}\right) = 1 + \frac{1}{192n^3} - \frac{1}{384n^4} + O\left(\frac{1}{n^5}\right).$$

For large n , this ratio is quite small.

For $n = 10^6$, the lower and upper bounds in (41) are

$$8.2639316883312400623566 \cdot 10^{5565708} \text{ and} \\ 8.2639316883312400623996 \cdot 10^{5565708}$$

Notice that the first 20 digits of the lower and upper bounds are the same. Therefore, we know that $n!$ begins these 20 digits. In fact, *Mathematica* can calculate $n!$, and the value is about $8.2639316883312400623766 \dots \cdot 10^{5565708}$.

(But note that if the lower and upper bounds of some x were 1.9 and 2.1, then *no* digits would agree, but we would still know that $x = 2 \pm 0.1$.)

We conclude that, using the bounds in (41), we can compute at least the first 10 significant digits of factorials of very large numbers.

Although the natural logarithms of the bounds above would be a little simpler, logs base 10 will most easily produce displayable values. Taking logs in (41), we have

$$n \lg\left(\frac{n}{e}\right) + \frac{\lg(2\pi n)}{2} + \frac{\lg(e)}{12n + \frac{3}{4n+2}} < \lg(n!) < n \lg\left(\frac{n}{e}\right) + \frac{\lg(2\pi n)}{2} + \frac{\lg(e)}{12n}.$$

So, for a given n , define $b_1(n)$ to be the lower bound of $\lg(n!)$:

$$b_1(n) = n \lg\left(\frac{n}{e}\right) + \frac{\lg(2\pi n)}{2} + \frac{\lg(e)}{12n + \frac{3}{4n+2}},$$

and define $b_2(n)$, as the upper bound of $\lg(n!)$:

$$b_2(n) = n \lg\left(\frac{n}{e}\right) + \frac{\lg(2\pi n)}{2} + \frac{\lg(e)}{12n}.$$

Our goal is to compute $\lg((n!)^2/(2n!))$. To get a lower bound, we use the lower bound for $\lg(n!)$ and the upper bound for $\lg((2n!))$. So, the lower bound for $\lg((n!)^2/(2n!))$ is

$$2b_1(n) - b_2(2n), \tag{42}$$

and the respective upper bound is

$$2b_2(n) - b_1(2n). \tag{43}$$

Now suppose we have computed $z \approx \lg(N)$ for some large N (for example, $N = n!$). To display the approximate value of $N = 10^z$, we need not compute 10^z , which would overflow if N is large enough. Instead, we can extract the mantissa and the exponent of z , and then display N in scientific notation. We have

$$p = \lfloor z \rfloor \tag{44}$$

$$m = 10^{z-p}. \tag{45}$$

We can now display N , the antilog of z , as $N = m \cdot 10^p$, where $1 \leq m < 10$. However, note that if the integer exponent p has d significant digits, then the mantissa m will have about d fewer significant digits than z has. This happens because the subtraction in (45) causes a loss of precision. For example, if $z = 100000.12$, then $10^z = 10^{0.12} \cdot 10^{100000} \approx 1.3 \cdot 10^{100000}$. Although $10^{0.12} \approx 1.3182567\dots$, only the first two digits in the mantissa of 10^z are meaningful. This is because all we know about this “0.12” is that it is a number between .115 and .125, and $10^{.115} \approx 1.303$ and $10^{.125} \approx 1.334$.

If $n = 10^6$, then we can estimate $\lg((n!)^2/(2n)!)$ as follows. The lower and upper bounds for \lg of $(n!)^2/(2n)!$ in (42) and (43) are

$$\begin{aligned} & -602056.74275297175655031271186 \quad \text{and} \\ & -602056.74275297175655031270705. \end{aligned}$$

These agree to 25 digits. If we extract the mantissa and the exponent for these two logarithms, we obtain, respectively,

$$\begin{aligned} & 1.8082023454706427717249 \cdot 10^{-602057} \quad \text{and} \\ & 1.8082023454706427717449 \cdot 10^{-602057}. \end{aligned}$$

These agree with each other to 20 digits. This implies that, to 20 significant digits, the value of $(n!)^2/(2n)!$ is $1.8082023454706427717 \cdot 10^{-602057}$. Notice that we lost several significant digits because the exponent has 6 digits.

$n = 10^6$ is small enough that *Mathematica* can compute $(n!)^2/(2n)!$ directly. The value is about $1.8082023454706427717343 \cdot 10^{-602057}$.

Here is a straightforward implementation of some of the above equations as *Mathematica* code:

```
lowerFactorial[n_] := (n/E)^n * Sqrt[2 Pi n] * Exp[1/(12 n + 3/(4 n + 2))]
upperFactorial[n_] := (n/E)^n * Sqrt[2 Pi n] * Exp[1/(12 n)]
log10LowerFactorial[n_] := n * Log[10, n/E] + Log[10, 2 Pi n]/2 + Log[10, E]/(12 n + 3/(4 n + 2))
log10UpperFactorial[n_] := n * Log[10, n/E] + Log[10, 2 Pi n]/2 + Log[10, E]/(12 n)
(* here are the lower and upper bounds of log10[ (n!)^2/(2n)! ] *)
log10LowerRatio[n_] := 2*log10LowerFactorial[n] - log10UpperFactorial[2 n]
log10UpperRatio[n_] := 2*log10UpperFactorial[n] - log10LowerFactorial[2 n]
(* log of lower and upper bounds of tiny *)
log10TinyLower[a0_, n_, s_] := n * Log[10, s - a0] - (2 n - 1) * Log[10, 2] - log10UpperRatio[n]
log10TinyUpper[a0_, n_, s_] := n * Log[10, s - a0] - (2 n - 1) * Log[10, 2] - log10LowerRatio[n]
getME1[x_] := { 10^(x - Floor[x]), Floor[x] } (* get matissa and exponent of antilog base 10 *)
```

Let's use this code to calculate $J_{68100151}(10)$. First, obtain $n = 68100151$ and the sum

$$s = 10 + 7.233082813117408154954409388818928756297 \cdot 10^{-9}$$

truncated from Table 3. (Or, one may compute s by adding 68100151 terms directly). Then, run the following *Mathematica* code:

```
a0 = 10
n = 68100151
s = 10 + 7.233082813117408154954409388818928756297 * 10^(-9)
logt1 = log10TinyLower[a0, n, s]
logt2 = log10TinyUpper[a0, n, s]
{m1, e1} = getME1[logt1]
{m2, e2} = getME1[logt2]
```

The results are

$$\begin{aligned} \{m1, e1\} &= \{9.6492736004286844634795529419197, -554381308\} \\ \{m2, e2\} &= \{9.6492736004286844634795532800687, -554381308\} \end{aligned}$$

Of the 32 significant digits in the mantissas, 24 of them agree. Therefore, we know that

$$t \approx 9.64927360042868446347955 \cdot 10^{-554381308},$$

where all 24 displayed digits in the mantissa are correct. This is consistent with the result

$$t \approx 9.649273600428684463479553 \dots \cdot 10^{-554381308},$$

given in Table 1.

Finally, the following *Mathematica* code will calculate t for $a_0 = 25$:

```
a0 = 25
n = 727750522131718025058
s = 25 + 1.1432548064658205122366981824665412932645\
8197624224895807362049 * 10^(-22)
logt1 = log10TinyLower[a0, n, s]
logt2 = log10TinyUpper[a0, n, s]
{m1, e1} = getME1[logt1]
{m2, e2} = getME1[logt2]
```

Both pairs $\{m1, e1\}$ and $\{m2, e2\}$ are

```
{2.7238486475282335616855631278993497716469,
-15968197862152240928105}
```

which means that

$$t \approx 2.7238486475282335616855631278993497716469 \cdot 10^{-15968197862152240928105}$$

where all 41 digits in the mantissa are correct.

The number of correct digits here is determined by how closely the lower and upper bounds of $n!$ agree. The limitation here is that we are stuck with whatever accuracy these approximations provide. The Euler-Maclaurin summation formula, described next, allows us to get as much accuracy as we want.

5.2. Estimating $n!$ With the Euler-Maclaurin Summation Formula.

We can write

$$\sigma(n) := \ln((2n)!) - 2 \ln(n!) = \sum_{k=1}^{2n} \ln k - 2 \sum_{k=1}^n \ln k.$$

To get a good estimate for $\sigma(n)$, we will use the exact sum of $m - 1$ initial terms. Therefore, we split the sums:

$$\sigma(n) = \left[\sum_{k=1}^{m-1} + \sum_{k=m}^{2n} - 2 \left(\sum_{k=1}^{m-1} + \sum_{k=m}^n \right) \right] \ln k = - \left(\sum_{k=1}^{m-1} + \sum_{k=m}^n - \sum_{k=n+1}^{2n} \right) \ln k. \quad (46)$$

It remains to estimate $\sum_{k=m}^n \ln k$ and $\sum_{k=n+1}^{2n} \ln k$. Therefore, we apply the Euler-Maclaurin summation formula (19) with

$$f(x) = \ln x.$$

The derivatives are given by

$$f^{(k)}(x) = \frac{(-1)^{k-1} (k-1)!}{x^k}, \quad k = 1, 2, \dots \quad (47)$$

Furthermore, we have

$$\int_m^n f(x) dx = \int_m^n \ln x dx = x(\ln x - 1) \Big|_m^n = n(\ln n - 1) - m(\ln m - 1).$$

This yields

$$\sum_{k=m}^n \ln k = \psi_\mu(m, n) + R_\mu^*(m, n)$$

with the approximation

$$\begin{aligned} \psi_\mu(m, n) = & n(\ln n - 1) - m(\ln m - 1) + \frac{\ln m + \ln n}{2} \\ & + \sum_{j=1}^{\mu} \frac{B_{2j}}{2j(2j-1)} \left(\frac{1}{n^{2j-1}} - \frac{1}{m^{2j-1}} \right) \end{aligned} \quad (48)$$

and the remainder term

$$R_\mu^*(m, n) = \frac{1}{2\mu+1} \sum_{k=m}^{n-1} \int_0^1 \frac{B_{2\mu+1}(x)}{(k+x)^{2\mu+1}} dx.$$

Since the absolute values of all derivatives in Equation (47) are strictly decreasing, we find the error estimate

$$|R_\mu^*(m, n)| < |\tilde{R}_\mu^*(m, n)|$$

with

$$\tilde{R}_\mu^*(m, n) = \frac{n-m}{2\mu+1} \int_0^1 \frac{B_{2\mu+1}(x)}{(m+x)^{2\mu+1}} dx. \quad (49)$$

It follows that the approximation for (46) is given by

$$\tilde{\sigma}_{m,\mu}(n) = - \sum_{k=1}^{m-1} \ln k - \psi_\mu(m, n) + \psi_\mu(n+1, 2n) \quad (50)$$

with the error bound

$$|\tilde{\sigma}_{m,\mu}(n) - \sigma(n)| < |\tilde{R}_\mu^*(m, n)| + |\tilde{R}_\mu^*(n+1, 2n)|. \quad (51)$$

5.3. Computing t .

We can now put all of this together to compute the value of t .

The *Mathematica* code to perform these calculations is in Appendix A.1.

There are three main parts of the code.

The first part, in module `getNValueAndSumForA0[]`, computes the smallest $n = n_0$ for which the sum of $a_1 + \dots + a_n$ exceeds a_0 . The value of the sum is also computed and saved.

The second part, in module `lnFactRatio[]`, computes the natural logarithm of $(2n)!/(n!)^2$ for the $n = n_0$ just we found.

The third part runs a loop over some range of values of a_0 . This loop calls the above modules and saves and prints the results.

Appendix A.2 displays the results of running the *Mathematica* code.

REFERENCES

- [1] Uwe Bäsel, *A remark concerning sinc integrals*, <http://arxiv.org/abs/1404.5413>, [math.CA] 22 April 2014.
- [2] Robert Baillie, *Advanced problem 6241*, American Mathematical Monthly, vol. 87 (June-July, 1980) pp. 496–498.
- [3] Robert Baillie, David Borwein, and Jonathan M. Borwein, *Surprising Sinc Sums and Integrals*, American Mathematical Monthly, vol. 115, no. 10, (December, 2008), pp. 888–901.
- [4] Hacène Belbachir and Abdelkader Khelladi, *On a sum involving powers of reciprocals of an arithmetical progression*, Annales Mathematicae et Informaticae, vol. 34, (2007), pp. 29–31, http://ami.ektf.hu/uploads/papers/finalpdf/AMI_34_from29to31.pdf.
- [5] R. P. Boas and H. Pollard, *Continuous analogues of series*, American Mathematical Monthly, Vol. 80, no. 1, (January, 1973), pp. 18–25.
- [6] David Borwein and Jonathan M. Borwein, *Some remarkable properties of sinc and related integrals*, Ramanujan Journal, **5** (2001) pp. 73–89.
- [7] Grigorii Mikhailovich Fichtenholz: *Differential- und Integralrechnung*, vol. II, VEB Deutscher Verlag der Wissenschaften, Berlin 1964.
- [8] Godfrey Harold Hardy: *The Integral $\int_0^\infty \frac{\sin x}{x} dx$* , The Mathematical Gazette, **5** (1909), pp. 98–103.
- [9] Julian Havil, *Gamma: Exploring Euler's Constant*, Princeton University Press, Princeton, New Jersey, 2003.
- [10] Konrad Knopp, *Theorie und Anwendung der unendlichen Reihen*, 6. Aufl., Springer, Berlin/Göttingen/Heidelberg/New York 1996.
(Translation: *Theory and Application of Infinite Series*, Dover Publications, New York, 1990.)
- [11] A. J. Maria, *A remark on Stirling's Formula*, American Mathematical Monthly, Vol. 72, No. 10 (December, 1965), pp. 1096–1098, <http://www.jstor.org/stable/2315957>.
- [12] Trygve Nagell, *Zahlentheoretische Notizen I–VI* (Videnskapsselskapets Skrifter. I. Mat.-naturv. Klasse. 1923. No. 13), Jacob Dybwad, Kristiana, 1924, <http://archive.org/stream/skrifterutgitavv1923chri#page/354/mode/2up>.
- [13] Paul J. Nahin, *Inside Interesting Integrals*, Springer, New York 2015, pp. 83–84.
- [14] Georg Pólya: *Berechnung eines bestimmten Integrals*, Mathematische Annalen, **74** (1913), pp. 204–212.
- [15] Reinhold Remmert: *Funktionentheorie 1*, 4. Aufl., Springer, Berlin/Heidelberg 1995.
- [16] Herbert Robbins, *A remark on Stirling's Formula*, American Mathematical Monthly, Vol. 62, No. 1, (January, 1955), pp. 26–29, <http://www.jstor.org/stable/2308012>.
- [17] Hanspeter Schmid, *Two curious integrals and a graphic proof*, Elemente der Mathematik, **69** (2014), pp. 11–17.
- [18] Eric Weisstein, *Stirling's Approximation* from MathWorld, A Wolfram Web Resource, <http://mathworld.wolfram.com/StirlingsApproximation.html>.
- [19] *Harmonic number*, Wikipedia, https://en.wikipedia.org/wiki/Harmonic_number.

Uwe Bäsel, HTWK Leipzig, Faculty of Mechanical and Energy Engineering, Leipzig, Germany, uwe.baesel@htwk-leipzig.de

Robert Baillie, State College, Pennsylvania, USA, rjbaille@friei.com

APPENDIX A. *Mathematica* CODE

This code has been tested, and works, in *Mathematica* versions 7, 8, and 9.

A.1. Code. Here is the *Mathematica* code to compute the integrals for $a_0 = 10$ through $a_0 = 25$.

This code uses the Euler-Maclaurin summation formula to compute the corresponding n for which

$$\sum_{k=1}^n a_k = \sum_{k=1}^n \frac{1}{2k-1}$$

first exceeds a_0 . The value of this sum is also computed.

For each a_0 and the corresponding (large) n , this code then uses the Euler-Maclaurin summation formula to compute logarithm of the ratio of factorials

$$\frac{(2n)!}{n!^2},$$

which occurs in Equation (38).

This code is presented in a format (for example, without page numbers) that the user can copy and paste directly into *Mathematica*.

```
(*
for a given a0, the integral for a0 is
(Pi/2) * (1 - t).
the code below computes the value of t, which will be very tiny.

the number of significant digits in t depends on the values of
m1, mu1, nDecimals1, m2, mu2, nDecimals2, and accGoal and workPrec.
for convenience, we set all of those values here.
*)

(* these are used to compute sums of ak, and the n value, given a0 *)
mu1 = 10; (* number of derivative terms to find n and the sum of ak *)
m1 = 100001; (* 1 + number of initial terms in sum of ak *)
nDecimals1 = 100; (* get the sum if ak to this many digits after the decimal point *)

(* accGoal and workPrec help get more accurate roots *)
accGoal1 = 20;
workPrec1 = 2*accGoal1;

(* these are used to compute factorials of large numbers *)
mu2 = 5; (* number of derivative terms to compute factorials *)
m2 = 100001; (* 1 + number of initial terms in sum of logs *)
nDecimals2 = 100; (* want the sum to this many digits after the decimal point *)

(* workPrec2 helps get a more accurate value for the integral in R2 *)
workPrec2 = 40;

(* the number of accurate digits in the result depends on these initial values *)
Print["m1 = ", m1, ", mu1 = ", mu1, ", nDecimals1 = ", nDecimals1,
      ", accGoal1 = ", accGoal1, ", workPrec1 = ", workPrec1];
Print["m2 = ", m2, ", mu2 = ", mu2, ", nDecimals2 = ", nDecimals2, ", workPrec2 = ", workPrec2];

(* define two utility functions, getME[ ] and removeQuestionableDigits[ ] *)
getME[c_] :=
Module[
  (* get matissa and exponent of c, an antilog base 10. *)
  { expo, diff, mant },
  expo = Floor[c];
```

```

diff = c - expo;
If[Accuracy[diff] < 2,
  mant = 1, (* not enough significant digits remain after subtraction *)
  mant = 10^diff
];
Return[ { mant , expo } ]
] (* end of Module *)

removeQuestionableDigits[c_, errorEst_] :=
Module[
(* c may have many digits that are significant as far as Mathematica is concerned,
  but the estimated error for c, based on an integral, might make some of those
  digits be meaningless.
  example: if
    c = 1.2345678901234567890123456789012345 * 10^-7 (35 digits),
  and the error estimate for c is 6.513 * 10^-20, then return
    c = 1.2345678901234567890123457*10^-7 (26 digits).
  to round c to have n digits after the decimal point, call
    removeQuestionableDigits[ c , 10^-(n+1) ] .
  note that this may introduce an error of up to 0.5 * 10^-(n+1) .
*)
{ digitsRightOfDP, log10c, mant, expo, numDigitsCorrect, power10ToRound },
digitsRightOfDP = Floor[ -Log[10, Abs[errorEst]] ];
If[digitsRightOfDP >= Floor[Accuracy[c]],
  Return[c] (* nothing to do; example: c = 1.2345, errorEst = 10^-20 *)
];
log10c = Log[10, Abs[c]];
{mant, expo} = getME[log10c]; (* c = mant * 10^expo *)
numDigitsCorrect = expo + digitsRightOfDP;
numDigitsCorrect = Max[numDigitsCorrect, 1];
power10ToRound = N[10^-numDigitsCorrect, numDigitsCorrect];
Return[ 10^expo * Round[mant, power10ToRound ] ]
] (* end of Module *)

```

(* here is the code to compute n for a given value of a0 *)

```

(* compute s(m1 - 1); see Equation (4) *)
initialAkSum = N[Sum[1/(2 k - 1), {k, 1, m1 - 1}], nDecimals1 + 10];
(*
we now have the sum to at least nDecimals1 digits, with essentially no roundoff error.
next, round it to have (nDecimals1) digits after the decimal point.
note that this sum might now have an error of 0.5*10^-(nDecimals1 + 1).
*)
initialAkSum = removeQuestionableDigits[initialAkSum, 10^-(nDecimals1 + 1)];
Print["sum of the first ", m1 - 1, " ak values = ", initialAkSum];

```

```

(* curlyPhi = Euler-Maclaurin sum without the error term; see Equation (23) *)
curlyPhi[mu_, m_, n_] :=
  1/2 (Log[2 n - 1] - Log[2 m - 1] + 1/(2 m - 1) + 1/(2 n - 1)) -

```

```
Sum[(2^(2 j - 1) BernoulliB[2 j])/(2 j) * (1/(2 n - 1)^(2 j) - 1/(2 m - 1)^(2 j)),
  {j, 1, mu}];
```

```
(* R from Equation (24) *)
```

```
R1[mu_, m_, n_] := -2^(2 mu + 1) (n - m) *
( NIntegrate[BernoulliB[2 mu + 1, x]/(2 (m + x) - 1)^(2 mu + 2), {x, 0, 1/2},
  WorkingPrecision -> workPrec1] +
  NIntegrate[BernoulliB[2 mu + 1, x]/(2 (m + x) - 1)^(2 mu + 2), {x, 1/2, 1},
  WorkingPrecision -> workPrec1] );
```

```
getNValueAndSumForA0[a0_, m_, mu_, initialAkSum_, accGoal_, workPrec_, iPrint_] :=
```

```
Module[
```

```
(* this module computes and returns three values:
```

- (1) the smallest n value for which the sum of 1/ak (k >= 1) first exceeds a0;
- (2) the corresponding sum; this is slightly larger than a0;
- (3) the absolute value of error estimate of the sum.

```
the error bound is based on equation (24). based on this error bound,
we discard any digits in the sum that might not be correct.
```

```
if iPrint == 1, this prints out various internal values for debugging.
```

```
*)
```

```
{ rootGuess = 10^6, root, r, nValue, errorEst, sum1Est, sum1Shortened,
  sum2Est, sum2Shortened },
```

```
sum1Est = sum1Shortened = sum2Est = sum2Shortened = 0;
```

```
root = n /. FindRoot[initialAkSum + curlyPhi[mu, m, n] == a0, {n, rootGuess},
  AccuracyGoal -> accGoal, WorkingPrecision -> workPrec];
```

```
(* check if the difference in r is greater than the error bound *)
```

```
If[R1[mu, m, root] > 0,
```

```
  r = Floor[root],
```

```
  r = Floor[root] + 1
```

```
];
```

```
errorEst = Abs[R1[mu, m, r]];
```

```
If[Abs[a0 - (initialAkSum + curlyPhi[mu, m, r])] > errorEst,
```

```
  nValue = Floor[root] + 1,
```

```
  nValue = 0 (* failed to find a valid root *)
```

```
];
```

```
If[iPrint == 1,
```

```
  (* R1[mu, m, r] and R1[mu, m, root] usually agree to about 9 significant digits *)
```

```
  Print["a0 = ", a0, ", root = ", root, ", nValue = ", nValue,
```

```
    ", R1 = ", N[R1[mu, m, root], 6],
```

```
    ", diff1 = ", N[a0 - (initialAkSum + curlyPhi[mu, m, r]), 10],
```

```
    ", diff2 = ", N[initialAkSum + curlyPhi[mu, m, r + 1] - a0, 10]
```

```
  ];
```

```
];
```

```
If[nValue > 0, (* we do not use sum1Est or sum1Shortened *)
```

```
  (* sum1Est = initialAkSum + curlyPhi[mu, m, nValue - 1]; *) (* a1 + a2 + ... + a[nValue-1] *)
```

```
  (* note: we know initialAkSum to only (nDecimals1) decimal places.
```

```
    therefore, we cannot know sum2Est to more decimal places than that.
```

```

        when Mathematica adds curlyPhi to initialAkSum, it will not retain
        more than (nDecimals1) decimal places in the result.
    *)
    sum2Est = initialAkSum + curlyPhi[mu, m, nValue]; (* a1 + a2 + ... + a[nValue] > a0 *)
    (* keep only those digits that are justified, given the error estimate from R1 *)
    (* sum1Shortened = removeQuestionableDigits[sum1Est, errorEst]; *)
    sum2Shortened = removeQuestionableDigits[sum2Est, errorEst];
    If[iPrint == 1,
        Print["a0 = ", a0, ", errorEst = ", N[errorEst, 6],
            ", curlyPhi = ", curlyPhi[mu, m, nValue],
            ", sum2Est = ", sum2Est, ", shortened sum2Est = ", sum2Shortened];
    ]
];

Return[ { nValue , sum2Shortened , N[errorEst, 6] } ]

] (* end Module *)

(*
here is the code to use the Euler-Maclaurin summation formula to compute factorials.
*)

(* this sum will have about 7 digits before the decimal point, so add 15 here *)
initialLogSum = N[Sum[Log[k], {k, 1, m2 - 1}], nDecimals2 + 15];
(*
we now have the sum to at least nDecimals2 digits, with essentially no roundoff error.
next, round it to have (nDecimals2) digits after the decimal point.
note that this sum might now have an error of  $0.5 \cdot 10^{-(nDecimals2 + 1)}$ .
*)
initialLogSum = removeQuestionableDigits[initialLogSum,  $10^{-(nDecimals2 + 1)}$ ];
Print["sum of the first ", m2 - 1, " logs = ", initialLogSum];

(* curlyPsi = Euler-Maclaurin sum without the error term, see Equation (48) *)
curlyPsi[mu_, m_, n_] :=
    n*(Log[n] - 1) - m*(Log[m] - 1) + (Log[m] + Log[n])/2 +
    Sum[BernoulliB[2 j]/(2 j (2 j - 1)) * (1/n^(2 j - 1) - 1/m^(2 j - 1)),
        {j, 1, mu}];

(* wide tilde R from Equation (49) *)
R2[mu_, m_, n_] := (n - m)/(2 mu + 1) *
    ( NIntegrate[BernoulliB[2 mu + 1, x]/(m + x)^(2 mu + 1), {x, 0, 1/2},
        WorkingPrecision -> workPrec2] +
    NIntegrate[BernoulliB[2 mu + 1, x]/(m + x)^(2 mu + 1), {x, 1/2, 1},
        WorkingPrecision -> workPrec2] );

(* Equation (51) *)
lnFactRatioErrorBound[mu_, m_, n_] := Abs[R2[mu, m, n]] + Abs[R2[mu, n + 1, 2 n]] ;

```

```

lnFactRatio[mu_, m_, n_, iPrint_] :=
Module[
  (* compute the natural log of the ratio of two factorials:
    ln[ (2n)!/(n!)^2 ]; see Equation (50).
    also, throw away digits that are not justified based on R2[mu, m, n].
  *)
  { lnFactRat, lnFactErrorEst, lnFactRatShortened },
  (* note: we know initialLogSum to only (nDecimals2) decimal places.
    therefore, we cannot know lnFactRat to more decimal places than that.
    when Mathematica adds curlyPsi to initialLogSum, it will not retain
    more than (nDecimals2) decimal places in the result.
  *)
  lnFactRat = -initialLogSum - curlyPsi[mu, m, n] + curlyPsi[mu, n + 1, 2 n];
  lnFactErrorEst = lnFactRatioErrorBound[mu, m, n];
  lnFactRatShortened = removeQuestionableDigits[lnFactRat, lnFactErrorEst];
  If[iPrint == 1,
    Print["n = ", n, ", lnFactRat = ", lnFactRat, ", lnFactErrorEst = ", N[lnFactErrorEst, 5],
      ", lnFactRatShortened = ", lnFactRatShortened]
  ];

  Return[ lnFactRatShortened ]
] (* end of Module *)

(* all necessary functions are now defined *)

iPrint = 0; (* set to 1 to print intermediate data *)

(* expr1 and const1 are used only to display the expression used in FindRoot;
  in the actual calculations, we use curlyPhi[ ].
*)
If[iPrint == 1,
  expr1 = Simplify[ curlyPhi[mu1, m1, x] ];
  const1 = Part[expr1, 1]; (* extract the rational number *)
  Print["approximate phi to be used in FindRoot: ", (expr1 - const1) + N[const1, nDecimals1]];
];

(* calculate everything we need for this range of a0 values *)
a0First = 10;
a0Last = 25;

Print["integral for a0 is (Pi/2) * (1 - t)"];

a0List = nList = sumAkList = r1List = mantList = expoList = {};
For[a0 = a0First, a0 <= a0Last, a0++,
  { n , sumAk , r1 } = getNValueAndSumForA0[a0, m1, mu1, initialAkSum, accGoal1, workPrec1, iPrint];
  If[n == 0, Break[] ]; (* could not compute a valid value of n or sum of ak *)
  (* compute the natural log of (2n)!/(n!)^2 *)
  lnFactorialRatio = lnFactRatio[mu2, m2, n, iPrint];
  (* compute log of t to base 10; see Equations (38) - (40) *)

```

```

logTBase10 = (n * Log[sumAk - a0] - (2 n - 1) * Log[2] + lnFactorialRatio) / Log[10];
{ mant , expo } = getME[ logTBase10 ];
(* save lists of these values in case we want to use them later *)
AppendTo[a0List, a0];
AppendTo[nList, n];
AppendTo[sumAkList, sumAk];
AppendTo[r1List, r1];
AppendTo[mantList, mant];
AppendTo[expoList, expo];
Print["a0 = ", a0, ", n = ", n, ", t = ", mant, " * 10^", expo];
] (* end For a0 loop *)

```

A.2. Results. Here are the values of the integrals for $a_0 = 10$ through $a_0 = 25$, obtained by running the code in Section A.1. The value of the integral for a_0 is $(\pi/2) \cdot (1 - t)$.

If a computed intermediate value had more digits after the decimal point than were justified by the corresponding error value (for example, Equation (24) or (51)), then the extra digits were discarded. Therefore, all digits shown below should be correct, rounded in the last decimal place.

```

integral for a0 = (Pi/2) * (1 - t)
a0 = 10, n = 68100151, t = 9.649273600428684463479553120939810530923242208735398 * 10^-554381308
a0 = 11, n = 503195829, t = 7.57929806494947536128349934756162195412431861759227 * 10^-4887781043
a0 = 12, n = 3718142208, t = 5.30200436015724605246826614752108917188558325098544 * 10^-39227165565
a0 = 13, n = 27473561358, t = 1.52739916984845667363367296109645541442392755493153 * 10^-297230209953
a0 = 14, n = 203003686106, t = 6.617345077783595182168242992545965700461478406777 * 10^-2419966945909
a0 = 15, n = 1500005624924, t = 5.26019597269976433379615815051550875066124252042 * 10^-18988869014266
a0 = 16, n = 11083625711271, t = 4.06751521421327233190115950829638686972451899823 * 10^-148452517153987
a0 = 17, n = 81897532160125, t = 2.8703074957720537216132995767534053015103162770 * 10^-1261337931785960
a0 = 18, n = 605145459495141, t = 1.46932966274512803735093876340436661798499994 * 10^-9192758406970262
a0 = 19, n = 4471453748222757, t = 7.36887339695623805028019042180415757921528157 * 10^-73134639260589997
a0 = 20, n = 33039822589391676, t = 5.8024461422390775663611817270349845938468954 * 10^-579426465025122292
a0 = 21, n = 244133102611731231, t = 1.3869021709986676325063938918439160007102035 * 10^-4427143349945912840
a0 = 22, n = 1803913190804074904, t = 4.10151245193385022941804060193305405447094 * 10^-34064698104956009918
a0 = 23, n = 13329215764452299411, t = 1.71715972357092319138607947022428968556534 * 10^-259489336406929338805
a0 = 24, n = 98490323038288832267, t = 1.4982758030623762036996263232893870122517 * 10^-2011250066953860707590
a0 = 25, n = 727750522131718025058, t = 2.723848647528233561685563127899349771647 * 10^-15968197862152240928105

```

The above code began by initializing these parameters:

```

m1 = 100001, mu1 = 10, nDecimals1 = 100, accGoal1 = 20, workPrec1 = 40
m2 = 100001, mu2 = 5, nDecimals2 = 100, workPrec2 = 40

```

The reader can obtain additional significant digits in t by increasing some of these parameters. Much of the time is already spent computing $s(m1 - 1)$, so it is more efficient to increase $\mu1$ or $\mu2$.

These calculations can also be extended beyond $a_0 = 25$. For example, running the code with $a0Last = 40$ with the above parameters gives, for $a_0 = 40$,

```

t = 1.8758610 * 10^-266134053348172015148849587491648267

```

As a_0 increases, more and more of the significant digits in the calculation are consumed in the exponent. Merely increasing $\mu1$ to 11 gives, for $a_0 = 40$,

```

t = 1.8758609814211976 * 10^-266134053348172015148849587491648267

```